§ 2 External Field Methods

§ 2.1 The Quantum Effective Action

Vacuum-vacuum amplitude in the presence of currents $\mathcal{J}_r(x)$:

$$Z[\mathcal{J}] \equiv \langle \text{VAC, out} | \text{VAC, in} \rangle_{\mathcal{J}} \qquad (1)$$

$$= \int \left[\prod_{s,y} d\phi^s(y) \right] \exp\left(iI[\phi] + i \int d^4x\, \phi^r(x) \mathcal{J}_r(x) + \varepsilon \text{ terms} \right)$$

the fields ϕ^r are arbitrary types of fields including fermionic type

$\longrightarrow$ Feynman diagrams now contain vertices to which a <u>single</u> ϕ^r-line is attached.

vertex : $i\mathcal{J}_r(x)$

Derivatives of (1) give

$$\left[\frac{\delta}{\delta \mathcal{J}_r(y)} Z(\mathcal{J}) \right]_{\mathcal{J}=0} = i \int \left[\prod_{r,x} d\phi^r(x) \right] \phi^r(y) \exp\left\{ iI[\phi] + \mathcal{O}(\varepsilon) \right\}$$

$$= i \langle \text{VAC, out} | \Phi^r(y) | \text{VAC, in} \rangle_{\mathcal{J}=0}$$

We have the expansion

$$Z[\mathcal{J}] = \sum_{N=0}^{\infty} \frac{1}{N!} \left(iW[\mathcal{J}] \right)^N = \exp\left(iW[\mathcal{J}] \right),$$

where $i W[J]$ is the sum of all connected vacuum–vacuum amplitudes

Define

$$\phi^r_J(x) \equiv \frac{\langle VAC, out | \Phi^r(x) | VAC, in \rangle_J}{\langle VAC, out | VAC, in \rangle_J} \qquad (2)$$

$$= \frac{-i}{Z[J]} \frac{\delta}{\delta J_r(x)} Z[J] = \frac{\delta}{\delta J_r(x)} W[J]$$

Suppose $\phi^r_J(x) = \phi^r(x)$ if $J_r(x) = J_{\phi r}(x)$

Definition:

The "quantum effective action" $\Gamma(\phi)$ is defined by the Legendre transformation

$$\Gamma[\phi] \equiv -\int d^4x \, \phi^r(x) J_{\phi r}(x) + W[J_\phi] \qquad (3)$$

Remark:

Note that

$$\frac{\delta \Gamma[\phi]}{\delta \phi^s(y)} = -\int d^4x \, \phi^r(x) \frac{\delta J_{\phi r}(x)}{\delta \phi^s(y)} - J_{\phi s}(y)$$
$$+ \int d^4x \left[\frac{\delta W[J]}{\delta J_r(x)} \right]_{J = J_\phi} \frac{\delta J_{\phi r}(x)}{\delta \phi^s(y)}$$

or using (2):

$$\frac{\delta \Gamma[\phi]}{\delta \phi^s(y)} = - J_{\phi s}(y)$$

Thus $\Gamma[\phi]$ is the "effective action" in the sense that the possible values for the external fields $\phi^s(y)$ in the <u>absence</u> of a current $\mathcal{J}$ are given by the stationary "points" of Γ:

$$\frac{\delta\Gamma[\phi]}{\delta\phi^s(y)} = 0 \quad \text{for } \mathcal{J} = 0 . \qquad (4)$$

$\rightarrow$ equation of motion for ϕ, taking quantum corrections into account

Let's replace $I[\phi]$ by $\Gamma[\phi]$ in (1) to see what happens:

$$\exp\left\{ i W_\Gamma[\mathcal{J}, g] \right\}$$

$$\equiv \int \prod_{r,x} d\phi^r(x) \exp\left\{ i g^{-1} \left[\Gamma[\phi] + \int d^4x\, \phi^r(x) \mathcal{J}_r(x) \right] + \mathcal{O}(\varepsilon) \right\} \qquad (5)$$

with arbitrary constant g.

$\rightarrow$ propagator proportional to g

vertices are proportional to $1/g$

$\rightarrow$ graph with V vertices and I internal lines is proportional to g^{I-V}.

For any connected graph the #loops is

$$L = I - V + 1$$

$\rightarrow$ L-loop term in $W_\Gamma[\mathcal{J}, g]$ has g dependence

$$\left(W_\Gamma[\mathcal{J}, g]\right)_{L \text{ loops}} \sim g^{L-1}$$

or equivalently

$$W_\Gamma[\mathcal{J}, g] = \sum_{L=0}^{\infty} g^{L-1} W_\Gamma^{(L)}[\mathcal{J}]$$

We are interested in sum of "tree" graphs (i.e. without loops)

$\longrightarrow W_\Gamma^{(0)}[\mathcal{J}]$, i.e. the $g \to 0$ limit of (5)

In this limit the path integral is dominated by the point of stationary phase:

$$\exp\left\{ i W_\Gamma[\mathcal{J}, g] \right\} \sim \exp\left\{ i g^{-1} \left[\Gamma[\phi_\mathcal{J}] + \int d^4x \, \phi^r_\mathcal{J}(x) \mathcal{J}_r(x) \right] \right\},$$

where $\left. \dfrac{\delta \Gamma[\phi]}{\delta \phi^r(x)} \right|_{\phi = \phi_\mathcal{J}} = - \mathcal{J}_r(x)$

$$\Rightarrow W_\Gamma^{(0)}[\mathcal{J}] = \underbrace{\Gamma[\phi_\mathcal{J}] + \int d^4x \, \phi^r_\mathcal{J}(x) \mathcal{J}_r(x)}_{\overset{(3)}{=} W[\mathcal{J}]}$$

This shows that $W[\mathcal{J}]$ may be calculated by using $\Gamma[\phi]$ in place of $I[\phi]$ and keeping only <u>tree</u> (0-loop) graphs!

In other words:

$$iW[\mathcal{J}] = \int_{\substack{\text{connected}\\ \text{tree}}} \left[\prod_{r,x} d\phi^r(x)\right] \exp\left\{ i\Gamma[\phi] + i\int \phi^r(x)\mathcal{J}_r(x)d^4x \right\}$$

Now, any connected graph for $iW[\mathcal{J}]$ can be regarded as a tree, whose vertices consist of "one-particle-irreducible" subgraphs.

"graph" which cannot be disconnected by cutting through lines

$\longrightarrow$ $i\Gamma[\phi]$ must be the sum of all one-particle-irreducible connected graphs with arbitrary number of external lines.

Equivalently, $i\Gamma[\phi_0]$ for some fixed $\phi_0^r(x)$ may be expressed as the sum of one-particle-irreducible graphs:

$$i\Gamma[\phi_0] = \int_{\substack{\text{1PI}\\ \text{connected}}} \left[\prod_{r,x} d\phi^r(x)\right] \exp\left\{ iI[\phi + \phi_0] \right\}$$

or $$\exp\left[i\Gamma[\phi_0]\right] = \int_{\text{1PI}} \left[\prod_{r,x} d\phi^r(x)\right] \exp\left\{ iI[\phi+\phi_0] \right\}$$

Formalism provides simple method for summing tree graphs:

$$\Delta^{rx,sy} \equiv \frac{\delta^2 W[J]}{\delta J_r(x)\,\delta J_s(y)} = \frac{\delta \phi^r(x)}{\delta J_s(y)}$$

$\uparrow$

complete 2-point fct.

$$\Pi_{rx,sy} \equiv \frac{\delta^2 \Gamma[\phi]}{\delta \phi^r(x)\,\delta \phi^s(y)} = - \frac{\delta J_{\phi r}(x)}{\delta \phi^s(x)}$$

$\uparrow$

1PI part of 2-point fct.

$\Longrightarrow$ the "matrices" Δ and Π are related by:

$$\Delta = -\Pi^{-1}$$